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MIXED FORMULATIONS OF BENDING PROBLEMS FOR HOMOGENEOUS ELASTIC PLATES AND BEAMS

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Mixed formulations of bending problems for homogeneous plates (beams) are proposed, whose essence is that the deformation of a plate (beam) near its fixed boundary is described by the three-dimensional elasticity equations, and the remaining part by the conventional equations of plate (beam) bending. At the interface between these regions, the solutions of these equations are joined. The mixed formulation allows one to describe the three-dimensional stress state in the neighborhood of the fixed boundaries of plates (beams) and take into account the complex nature of the fixing conditions. Finite-element implementation is more efficient for the mixed formulations of plate (beam) bending problems than for the well-known three-dimensional formulations.

Key words: *homogeneous plates and beams, three-dimensional elastic problem, Kirchhoff and Reissner theories.*

Introduction. It is well known [1–5] that plate and beam bending problems are commonly formulated using hypotheses that impose certain restrictions on the displacement, strain, and stress fields and introduce unremovable errors into the solutions. Moreover, the existing theories of plate (beam) bending ignore the complex nature of their fixing conditions, for example, in the case of a plate (beam) with a partly clamped edge. Drawbacks and advantages of various formulations of plate bending problems are discussed in [3–7]. Three-dimensional discrete basic models of plates (beams) that take into account any fixing conditions and provide for specified solution accuracy have large dimensions.

In the present paper, we consider mixed formulations of bending problems for elastic homogeneous plates and beams [8], whose essence is as follows. A plate (beam) is treated as a three-dimensional body in the neighborhood of its fixed boundary, and the deformation in this region is described using the three-dimensional elasticity equations. The deformation of the remaining part of the plate (beam) is described by the equations of a Reissner plate [5] (a Kirchhoff beam). At the interface between these regions, the solutions of the two problems are joined.

The mixed formulations of bending problems have the following advantages. First, they describe the three-dimensional stress state in the neighborhood of the fixed boundaries of plates (beams), which allows one to take into account the complex nature of the fixing conditions. Second, varying certain geometric parameters that appear in mixed formulations, one can construct a mixed discrete model of a plate (beam) in which the stresses in the neighborhood of the fixed boundary differ from those in the basic model by a specified small quantity. Third, finite-element implementation [9, 10] for the mixed discrete models of plates (beams) requires less computer time and memory than that for the basic models.

1. Mixed Formulations of Plate and Beam Bending Problems. 1.1. We consider an isotropic homogeneous linear-elastic plate of constant thickness which occupies a region V in Cartesian coordinates xyz. The middle plane of the plate coincides with the xOy plane. The plate is loaded by surface forces q_z and is fixed on the boundary S_r. We denote the neighborhood of the boundary S_r by V_r. The region V_r can be treated as a set of spheres of radius $R_r \geqslant C_r$, whose centers are points of the boundary S_r. Calculations show that it is

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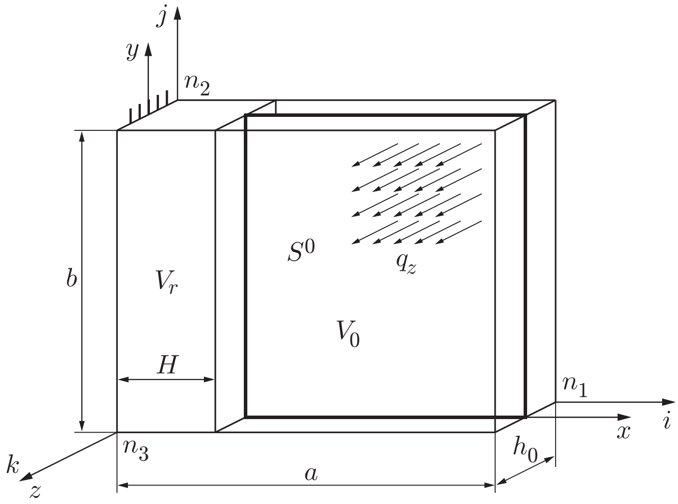

Fig. 1

expedient to use the values of $C_r \geqslant 3h_0$ (h_0 is the plate thickness). The shape of the region V_r is chosen so as to make calculations convenient. We introduce the following notation: u^r, v^r, and w^r and u^0, v^0, and w^0 are the displacement functions of the plate in the regions V_r and V_0, respectively, and S_H is the interface between the regions V_r and V_0 ($V_0 = V - V_r$). For simplicity, we consider the mixed formulation of the problem for a $a \times b$ rectangular plate (i.e., $V = a \times b \times h_0$) which is partly clamped for $x = 0$. In Fig. 1, the clamped part of the plate is shown by hatching and $n_1 = 81$, $n_2 = 51$, $n_3 = 11$, $a = 80h$, $b = 50h$, and $h_0 = 10h$. In this case, the boundary S_H is the intersection of the plate and the plane $x = H$. If the plate V is in equilibrium, the following conditions hold on the boundary S_H for $x = H$:

$$u^r = u^0, \qquad v^r = v^0, \qquad w^r = w^0; \tag{1}$$

$$\sigma_x^r = \sigma_x^0, \qquad \tau_{xy}^r = \tau_{xy}^0; \tag{2}$$

$$\tau_{xz}^r = \tau_{xz}^0. \tag{3}$$

Here σ_x^r, τ_{xz}^r, and τ_{xy}^r and σ_x^0, τ_{xz}^0, and τ_{xy}^0 are the stresses acting on the boundary S_H in the regions V_r and V_0, respectively.

For the region V_r, we formulate the following three-dimensional elastic problem:

$$A(\boldsymbol{u}_r) = \boldsymbol{p} \quad \text{in} \quad V_r; \tag{4}$$

$$B(\boldsymbol{u}_r) = \boldsymbol{q}_r \quad \text{on} \quad S_q^r, \qquad u^r = v^r = w^r = 0 \quad \text{on} \quad S_r. \tag{5}$$

Here A is the equilibrium-equation operator, B is the operator of the static boundary conditions, $\boldsymbol{u}_r = \{u^r, v^r, w^r\}^{\mathrm{t}}$, $\boldsymbol{p} = \{0,0,0\}^{\mathrm{t}}$ is the body force vector in the region V_r, $\boldsymbol{q}_r = \{0,0,q_z\}^{\mathrm{t}}$ is the surface load vector in the region V_r, S_q^r is the boundary of the region V_r on which the loads are specified, and $S^r = S_H + S_r + S_q^r$ is the boundary of the region V_r. The conditions on the boundary S_H are given below.

1.2. We consider the region V_0 as a thin plate S^0 (in Fig. 1, the boundary of the middle surface of the plate S^0 is shown by a thick line). In the region V_0m, the plate bending problem using Reissner's theory is formulated as follows:

$$\Delta\Delta\varphi = q_z^0/D, \qquad \Delta\psi - k^2\psi = 0; \tag{6}$$

$$y = 0, b\colon \quad M_y = M_{xy} = Q_y = 0, \qquad x = a\colon \quad M_x = M_{xy} = Q_x = 0. \tag{7}$$

Here $\varphi(x,y)$ and $\psi(x,y)$ are unknown functions, $D = Eh_0^3/(12(1-\nu^2))$, E is Young's modulus, ν is Poisson's ratio, $q_z^0 = q_z(x,y,h_0/2)$ is the surface load, Δ is the Laplace operator, $k^2 = 2C/[D(1-\nu)]$, $C = Gh_0$, G is the shear modulus, M_x and M_y are the bending moments, M_{xy} is the twisting moment, and Q_x and Q_y are the transverse shear forces expressed in terms of φ and ψ.

The angles $\theta_x(x,y)$ and $\theta_y(x,y)$ of rotation of the normal to the middle plane of the plate S^0 and the plate deflection $w_0(x,y)$ are written in terms of φ and ψ:

$$\theta_x = -\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \qquad \theta_y = -\frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}, \qquad w_0 = \varphi - \frac{D}{C}\,\Delta\varphi. \tag{8}$$

According to Reissner's theory, the displacements u^0, v^0, and w^0 in the entire region V_0 and on the boundary S_H are approximated by the relations [2]

$$u^0 = z\theta_x(x,y), \qquad v^0 = z\theta_y(x,y), \qquad w^0 = w_0(x,y) \qquad \forall x,y,z \in S_H. \tag{9}$$

Inserting (9) into (1), we obtain

$$u^r = z\theta_x(x,y), \qquad v^r = z\theta_y(x,y), \qquad w^r = w_0(x,y) \qquad \forall x,y,z \in S_H. \tag{10}$$

It should be noted that conditions (10) satisfied on the boundary S_H do not imply that the functions u^r, v^r, and w^r satisfy conditions (10) in the entire region V_r. By virtue of (10), on the boundary S_H for $x = H$, the functions u^r, v^r, and w^r are written as

$$u^r = z\,\frac{2u^r(H,y,h_0/2)}{h_0}, \quad v^r = z\,\frac{2v^r(H,y,h_0/2)}{h_0}, \quad w^r = w^r(H,y,0) \quad \forall y,z \in S_H. \tag{11}$$

Substitution of (11) into (10) yields

$$u^r(H,y,h_0/2) = \frac{h_0}{2}\,\theta_x(H,y), \quad v^r(H,y,h_0/2) = \frac{h_0}{2}\,\theta_y(H,y),$$
$$w^r(H,y,0) = w_0(H,y) \qquad \forall y \in S_H. \tag{12}$$

Relations (8) and (12) can be combined to give

$$u^r\Big(H,y,\frac{h_0}{2}\Big) = \frac{h_0}{2}\Big(-\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\Big)\Big|_{x,y\in S_H}, \quad v^r\Big(H,y,\frac{h_0}{2}\Big) = \frac{h_0}{2}\Big(-\frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}\Big)\Big|_{x,y\in S_H},$$
$$w^r(H,y,0) = \varphi(x,y) - (D/C)\Delta\varphi\Big|_{x,y\in S_H} \qquad \forall y \in S_H. \tag{13}$$

In this case, for $x = H$, conditions (2) and (3) become

$$M_x^r = M_x^0, \qquad M_{xy}^r = M_{xy}^0; \tag{14}$$

$$Q_x^r = Q_x^0. \tag{15}$$

Here M_x^0, M_{xy}^0, and Q_x^0 are the bending moment, twisting moment, and transverse shear force of Reissner's plate S^0, respectively, expressed in terms of the functions ψ and φ [5] and M_x^r, M_{xy}^r, and Q_x^r are expressed in terms of u^r, v^r, and w^r and are calculated by the formulas

$$M_x^r = \int_{-h_0/2}^{h_0/2} z\,\sigma_x^r\,dz, \qquad M_{xy}^r = \int_{-h_0/2}^{h_0/2} z\,\tau_{xy}^r\,dz, \qquad Q_x^r = \int_{-h_0/2}^{h_0/2} \tau_{xz}^r\,dz.$$

Indeed, it follows from (9) and (10) that the displacement functions u^r, v^r, w^r, u^0, v^0, and w^0 on the boundary S_H in the regions V_r and V_0, respectively, correspond to the displacement approximations in Reissner's theory [2, 5]. Consequently, the stresses σ_x^r, τ_{xz}^r, τ_{xy}^r, σ_x^0, τ_{xz}^0, and τ_{xy}^0 on the boundary S_H are calculated in accordance with this theory as follows:

$$\sigma_x^r = z\,\frac{12}{h_0^3}\,M_x^r, \quad \tau_{xy}^r = z\,\frac{12}{h_0^3}\,M_{xy}^r, \quad \sigma_x^0 = z\,\frac{12}{h_0^3}\,M_x^0, \quad \tau_{xy}^0 = z\,\frac{12}{h_0^3}\,M_{xy}^0. \tag{16}$$

Taking into account (16), we infer that conditions (14) are equivalent to conditions (2). Condition (15) is obtained by integrating equality (3) with respect to z. Thus, condition (3) holds in the integral sense: according to Reissner's theory, the boundary conditions for the shear stresses τ_{xz} are expressed only in terms of the transverse shear forces. We note that conditions (3) and (15) are statically equivalent [5].

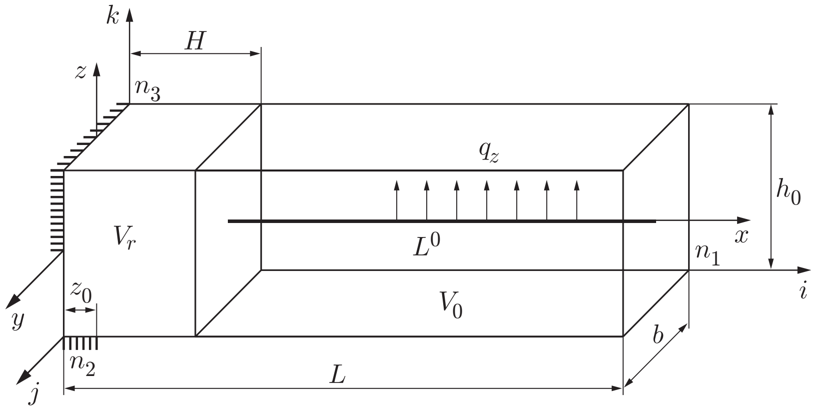

Fig. 2

It is shown that the mixed formulation of the plate bending problem reduces to Eqs. (4) and (6) subject to boundary conditions (5) and (7), conditions for displacements (11), and the joining conditions for the solutions u^r, v^r, w^r, ψ, and φ on the boundary S_H, i.e., conditions (13)–(15).

1.3. We consider an isotropic homogeneous linear-elastic beam which occupies a region V in Cartesian coordinates xyz. The beam axis coincides with the Ox axis, and the planes xOy and xOz are the horizontal and vertical planes of geometrical symmetry of the beam, respectively. The beam is loaded by forces q_z such that $q_z(x, y, z) = q_z(x, -y, z)$; i.e., the beam is bent in the vertical plane zOx. We consider the mixed formulation of the beam bending problem for a prismatic beam $V = L \times b \times h_0$ (Fig. 2). On the boundary S_r, the following complex constraints are imposed: for $x = 0$, the beam is partly clamped at its end and on the horizontal support. In Fig. 2, the fixed part of the beam is shown by hatching, $n_1 = 145$, $n_2 = 13$, $n_3 = 19$, $L = 144h$, $b = 12h$, $h_0 = 18h$, and $z_0 = 4h$. We denote the neighborhood of the boundary S_r by V_r. Using a similar line of reasoning as in Sec. 1.1, we require that equalities (1)–(3) should hold in the regions V_r and V_0 ($V_0 = V - V_r$). For the region V_r, we obtain a three-dimensional elastic problem in the form of (4), (5). The regions V_r and V_0 are separated by the plane $x = H$. Calculations show that it is expedient to use values of $C_r \geqslant 2.5h_0$ (h_0 is the characteristic dimension of the beam cross section) and supplement Eqs. (4) and (5) for the displacements v^r and w^r in the region V_r by the conditions

$$v^r(x, y, z) = 0, \qquad w^r(x, y, z) = w^r(x, 0, 0) \qquad \forall y, z \in V_r, \quad x_1 \leqslant x \leqslant H, \tag{17}$$

where $H \geqslant 2.5h_0 + z_0$ and $x_1 \geqslant H - 0.5h_0$.

We consider the region V_0 as a Kirchhoff beam L^0 (in Fig. 2, the axis of the beam L^0 is shown by a thick line). In this region, we formulate the beam bending problem [11]

$$\frac{\partial^4 w_0(x)}{\partial x^4} = \frac{q_z^0(x)}{EI_y}; \tag{18}$$

$$x = L: \qquad M = Q = 0, \tag{19}$$

where $w_0(x)$ is the beam deflection, I_y is the cross-sectional moment of inertia about the Oy axis, $q_z^0(x) = \int q_z(x, y, h_0/2)\, dy$ is the load, M is the bending moment, and Q is the transverse shear force.

The displacements of the three-dimensional beam L^0 are given by

$$u^0(x, y, z) = -z\,\frac{\partial w_0}{\partial x}, \qquad v^0(x, y, z) = 0, \qquad w^0(x, y, z) = w_0(x). \tag{20}$$

We note that, for the beam L^0, the common boundary S_H of the regions V_r and V_0 degenerates into the point $x = H$. Substituting relations (20) for $x = H$ into (1), we obtain the displacements on S_H:

$$u^r = -z\,\frac{\partial w_0(x)}{\partial x}\Big|_{x=H}, \qquad v^r\Big|_{x=H} = 0, \qquad w^r = w_0(H). \tag{21}$$

It should be noted that the condition $v^r\Big|_{x=H} = 0$ is satisfied by virtue of (17). Since u^r is independent of y, we impose the following constraint on the displacement u^r:

$$u^r(H, y, z) = u^r(H, 0, z) \qquad \forall y, z \in S_H. \tag{22}$$

Using (21) and (22), we write the functions u^r and w^r on S_H as

$$u^r = 2zu^r(H, 0, h_0/2)/h_0, \qquad w^r = w^r(H, 0, 0) \qquad \forall y, z \in S_H. \tag{23}$$

From (23) and (21) it follows that

$$u^r(H, 0, h_0/2) = -\frac{h_0}{2}\frac{\partial w_0}{\partial x}\Big|_{x=H}, \qquad w^r(H, 0, 0) = w_0(H) \qquad \text{for} \quad x = H. \tag{24}$$

Using (17), (22), and (23), as in Sec. 1.2, we replace conditions (2) and (3) by

$$M_x^r = M_x^0, \qquad Q_x^r = Q_x^0, \qquad \tau_{xy}^r = 0 \qquad \text{for} \quad x = H. \tag{25}$$

Here M_x^0 and Q_x^0 are the bending moment and transverse shear force of the beam L^0, respectively, which depend on its deflection w_0 [11] and M_x^r, Q_x^r, and τ_{xy}^r are functions of the displacements u^r, v^r, and w^r of the region V_r which, for $x = H$, are given by

$$M_x^r = \int\limits_S z\,\sigma_x^r\,dS, \qquad Q_x^r = \int\limits_S \tau_{xz}^r\,dS$$

(S is the cross section of the beam).

It is shown that the mixed formulation of the beam bending problem (see Fig. 2) reduces to Eqs. (4) and (18) subject to boundary conditions (5) and (19), the conditions for the displacements (17), (22), and (23), and the conditions (24) and (25) of joining of the solutions u^r, v^r, w^r, and w_0 on the boundary S_H. We note that the condition $v^r\Big|_{x=H} = 0$ is satisfied by virtue of (17).

The mixed formulations of the problems contain differential operators of the three-dimensional elastic problem and the plate or beam bending problem, which are known to be positive definite. Therefore, to solve the plate (beam) bending problems, one can use the finite-element method (FEM) (in the form of the Ritz method). In this case, it suffices to impose only kinematic boundary conditions; satisfaction of the conditions for the displacements [conditions (11) and (12) for plates and (17) and (22)–(24) for beams] presents no difficulties for discrete mixed models of plates (beams).

Remark 1. The mixed formulations of the bending problems contain geometrical parameters, i.e., the dimensions of the region V_r (in this case, H) which allow one to control the error of the solutions (see Sec. 2.1).

Remark 2. It is known [5] that Reissner's and Kirchhoff's theories give close results in describing the bending of thin plates at a distance from the fixed boundary (see Fig. 1, $x \geqslant H$). Therefore, to simplify the calculations for the discrete model of the plate S^0, we use Clough's finite element with the following nodal parameters [9]: w_0, θ_x, and θ_y ($\theta_x = \partial w_0/\partial x$, $\theta_y = \partial w_0/\partial y$, and w_0 is the plate deflection).

We point out some special features of Clough's finite elements (FEs). On the one hand, the approximating functions and the expression for the potential energy of the Clough's finite element are constructed using Kirchhoff's theory [9, 10]. On the other hand, the governing equations for this FE are obtained by varying the independent nodal parameters θ_x, θ_y, and w_0. Consequently, at the nodes of the Clough's FE, the statement of Reissner's theory holds that states that the functions θ_x and θ_y are independent of the deflection function w_0 [2]. To construct the global system of FEM equations, we require that the statement of Reissner's theory be satisfied at all nodes of the discrete model of the plate. Thus, the Clough's FE and the corresponding discrete model of the plate are constructed using the relations of Kirchhoff's and Reissner's plate bending theories. We note that Clough's finite elements allow one to satisfy conditions (12) in discrete form.

Remark 3. To construct a discrete model of the beam L^0, one should use the Hermitian finite element of the third order [10] with nodal parameters w_0 and dw_0/dx (w_0 is the beam deflection), which allow one to satisfy conditions (24) in discrete form.

2. Results of Numerical Experiments. 2.1. We consider an isotropic linear-elastic plate which occupies a region $V = 80h \times 50h \times 10h$ in the coordinate system xyz (see Fig. 1). On the boundary S_r: $\{x = 0,\ 0 \leqslant y \leqslant 50h,\ -3h \leqslant z \leqslant 3h\}$, the conditions $u = v = w = 0$ are specified, i.e., the plate is partly clamped at $x = 0$. The

TABLE 1

i	w_0	w_h	
		$H = 30h$	$H = 10h$
11	3.456	34.532	35.589
21	106.933	107.225	116.344
31	208.807	211.083	225.604
41	332.318	337.959	356.186
51	470.066	479.006	500.962
61	614.304	627.025	652.748
71	758.523	776.164	805.678
81	900.366	923.976	957.289

TABLE 2

x	$z = -0.5h$		$z = -2.5h$		$z = -3.5h$			$z = -4.5h$	
	σ_0	σ_h	σ_0	σ_h	σ_0	σ_h		σ_0	σ_h
						$H = 30h$	$H = 10h$		
$0.5h$	2.4673	2.4752	12.6614	12.6810	14.3000	14.3191	14.2407	1.2533	1.2549
$2.5h$	3.4659	3.4756	8.5684	8.5828	9.1465	9.1612	9.1099	7.0381	7.0499
$4.5h$	1.7131	1.7187	5.6871	5.6981	6.9251	6.9387	6.8854	8.0713	8.0878
$6.5h$	1.0908	1.0948	4.6617	4.6730	6.3581	6.3735	6.0640	8.2004	8.2203
$8.5h$	1.0488	1.0535	4.3828	4.3960	6.1437	6.1619	4.4740	8.0058	8.0294

three-dimensional (basic) model of the plate consists of first-order finite elements V_e^h shaped like a cube with side h and generates a $81 \times 51 \times 11$ grid, whose nodes are considered in the integer coordinate system ijk. The mixed (discrete) model of the plate is constructed for $H = 30h$ (i.e., $H = 3h_0$, where h_0 is the plate thickness). The region V_r is discretized into finite elements V_e^h. The region V_0 is treated as a thin plate S^0 (in Fig. 1, the boundary of its middle plane is shown by a thick line). The plate S^0 is partitioned into square Clough's finite elements [9] with side h and generates a square grid S_h^0, whose nodes have coordinates $(i, j, 6)$ ($i = 31, \ldots, 81$ and $j = 1, \ldots, 51$). Loads $q_z = 0.0324$ are applied to the nodes $(i, j, 6)$ of the grid S_h^0 ($i = 55, 60, 65, 70, 75$; $j = 30, 35, 40, 45$). The Young's modulus of the plate is $E = 1$, and the Poisson's ratio is $\nu = 0.3$, and $h = 0.5$. The calculations were performed for $H = h_0$ and $H = 3h_0$. For $H = 3h_0$, the displacements w_h (plate deflection) of the mixed model differ from the deflections w_0 of the basic model by not more than 2.5%. The values of w_0 and w_h ($j = 51$ and $k = 6$) are given in Table 1. Table 2 summarizes the equivalent stresses σ_h (for the mixed model) and σ_0 (for the basic model) calculated at the centroids of the finite elements of V_e^h ($y = 49.5h$, $H = 10h$, and $H = 30h$) according to the fourth strength theory. The maximum value of the stress σ_h differs from σ_0 by 0.06%. For $H = h_0$, the maximum value of w_h differs from w_0 by 6.3% and the maximum value of σ_h differs from σ_0 by 0.4%; i.e., the error of the solution decreases as H increases.

The basic model of the plate has 135,252 nodal unknowns, and the band width of the system of FEM equations is equal to 1722. The mixed model (for $H = 3h_0$) has 57,630 unknowns, and the band width is equal to 1749 and requires 2.3 times less computer memory compared to that of the basic model. The computation time for the mixed discrete plate model is a factor of 2.5 smaller than that for the basic model.

2.2. We consider an isotropic homogeneous linear-elastic beam which occupies a region $V = 144h \times 12h \times 18h$ in the coordinate system xyz (see Fig. 2). The conditions $u = v = w = 0$ are specified on the boundary S_r: $\{x = 0, -6h \leqslant y \leqslant 6h, 0 \leqslant z \leqslant 9h\} \cup \{0 \leqslant x \leqslant 4h, -6h \leqslant y \leqslant 6h, z = -h_0/2\}$; i.e., the beam is clamped on the horizontal support and is partly clamped at the end. The three-dimensional discrete (basic) model of the beam consists of finite elements V_e^h (see Sec. 2.1) and generates a $145 \times 13 \times 19$ grid, whose nodes are considered in the integer coordinate system ijk. The mixed (discrete) model of the beam is constructed for $H = 49h$ ($H = 2.5h_0 + 4h$), $x_1 = 36h$ ($x_1 = 2h_0$), and $C_r = 2.5h_0$. The region V_r is discretized into finite elements V_e^h. We consider the region V_0 as the beam L^0 (in Fig. 2, the axis of the beam L^0 is shown by a thick line). The beam L^0 is modeled by third-order Hermitian finite elements [10] of length h. In the coordinate system ijk, the nodes of this grid have coordinates $(i, 7, 10)$ ($i = 50, 51, \ldots, 145$). The forces $q_z = 0.0324$ are applied to the nodes with coordinates $(i, 7, 10)$ [$i = 49 + 12(k-1)$; $k = 1, \ldots, 7$]. The Young's modulus of the beam is $E = 1$, and the Poisson's ratio is $\nu = 0.3$, and

TABLE 3

i	w_0	w_h	i	w_0	w_h
13	2.053	2.051	109	102.772	105.585
37	16.898	16.909	133	134.449	139.009
61	41.581	41.932	145	150.373	155.729
85	71.172	72.649			

TABLE 4

x	$z = 8.5h$		$z = 6.5h$		$z = -1.5h$		$z = -6.5h$		$z = -8.5h$	
	σ_0	σ_h	σ_0	σ_h	σ_0	σ_h	σ_0	σ_h	σ_0	σ_h
$0.5h$	2.1425	2.1713	1.4028	1.4253	1.2698	1.2331	0.6316	0.6251	1.8986	1.8636
$2.5h$	1.9560	1.9778	1.4809	1.5035	0.6754	0.6509	0.7513	0.7364	3.4151	3.3622
$4.5h$	1.9402	1.9610	1.4779	1.5006	0.3619	0.3481	1.1586	1.1394	3.3805	3.3363
$6.5h$	1.9532	1.9744	1.4782	1.5008	0.1418	0.1397	1.4765	1.4548	2.3911	2.3626
$9.5h$	1.9458	1.9675	1.4765	1.4989	0.1570	0.1577	1.5207	1.4986	2.0392	2.0161
$15.5h$	1.8358	1.8578	1.4021	1.4243	0.2514	0.2435	1.4087	1.3865	1.8241	1.8022

$h = 0.111$. The maximum value of the deflection w_h of the mixed model differs from the deflections w_0 of the basic model of the beam by 3.2%. Table 3 presents the values of the deflections w_0 and w_h ($j = 7$ and $k = 10$). Table 4 ($y = -5.5h$) compares the equivalent stresses σ_h (for the mixed model) and σ_0 (for the basic model) calculated at the centroids of the finite elements V_e^h according to the fourth strength theory. In the neighborhood of the clamped part of the beam, the stresses σ_h differ from σ_0 by not more than 1.5%.

The basic model of the beam contains 106,899 unknowns, and the band width of the FE system of equations is equal to 786. The mixed model of the beam has 28,679 unknowns, and the band width equal to 866 occupies 3.4 times less computer memory compared to that of the basic model. The computation time for the mixed discrete model of the beam is four times smaller than that for the basic model.

REFERENCES

1. V. I. Samul', *Fundamentals of the Theory of Elasticity and Plasticity* [in Russian], Vysshaya Shkola, Moscow (1970).
2. V. V. Vasil'ev, "Classical theory of plates — history and state-of-the-art," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 3, 46–58 (1998).
3. A. L. Gol'denveizer, "Approximate methods for calculating thin elastic shells and plates," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 3, 134–149 (1997).
4. V. V. Vasil'ev, "Discussion of classical plate theory," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 4, 140–150 (1995).
5. V. V. Vasil'ev, "On thin-plate theory," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 3, 26–47 (1992).
6. P. A. Zhilin, "Poisson's and Kirchhoff's plate theories form the viewpoint of modern plate theory," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 3, 134–150 (1992).
7. N. A. Alfutov, "Some paradoxes of elastic thin plate theory," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 3, 65–72 (1992).
8. A. D. Matveev, "New three-dimensional formulations of applied problems of elasticity," Krasnoyarsk (2001). Deposited at VINITI 09.28.01, No. 2060-B2001.
9. P. M. Varvak, I. M. Buzun, A. S. Gorodetskii, et al., *Finite Element Method* [in Russian], Vishcha Shkola, Kiev (1981).
10. V. A. Postnov, *Numerical Methods of Ship Structural Analysis* [in Russian], Sudostroenie, Leningrad (1977).
11. N. M. Belyaev, *Strength of Materials* [in Russian], Nauka, Moscow (1976).